EXTREME POINTS AND THE KREIN–MILMAN THEOREM: A NOTE ON BREZIS PROBLEM 1

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Abstract. We organize a complete solution to Problem 1 from Brezis' *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* The argument is arranged as a short expository note: *we begin with elementary characterizations of extreme points, prove that every nonempty extreme subset of a compact convex set contains an extreme point.* Thus, we successful derive the Krein–Milman theorem. Finally, we conclude with explicit descriptions of the extreme points of the closed unit ball in several classical Banach spaces.

1. Introduction

The Krein–Milman theorem [Rudin, 1991] is one of the structural results at the heart of convexity in functional analysis: a compact convex set in a locally convex setting is generated, after taking closed convex hulls, by its extreme points. Problem 1 in Brezis' text isolates the main ideas behind this theorem and turns them into a sequence of manageable lemmas [Brezis, 2011].

This note follows the order of the Problem 1 in *Brezis' Functional Analysis book*, but we reorganize it into a proof-driven narrative. We first collect the basic equivalences for extreme points and extreme sets. We then prove that every nonempty extreme subset of a compact convex set contains an extreme point; this is the key step that allows one to deduce the Krein–Milman theorem by a separation argument. Finally, we compute the extreme points of the closed unit ball in a number of standard Banach spaces, thereby illustrating how geometry depends strongly on the ambient norm.

2. Preliminaries

Zorn's lemma plays an important role in many applications in analysis. It is the basic tool in proving some seemingly innocent existence statement.[Brezis, 2011]. In this note, we shall not discuss more about derivation and we just recall it in the Lemma below:

Lemma 2.1. *Every non-empty ordered set that is inductive has a maximal element.*

Let E be a normed vector space and let $K \subset E$ be convex.

Definition 2.2. A point $a \in K$ is an *extreme point* of K if whenever:

$$a = tx + (1-t)y, \qquad t \in (0,1), \quad x, y \in K, \tag{1}$$

one necessarily has $x = y = a$.

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When K is assumed compact and convex, a nonempty closed subset $M \subset K$ is called an *extreme set* if for every $t \in (0,1)$, we have:

$$tx + (1-t)y \in M, \quad x, y \in K \implies x, y \in M. \tag{2}$$

Thus, extreme sets are the closed faces of K.

We shall also use the Hahn–Banach theorem [Brezis, 2011, Rudin, 1991]. The Hahn-Banach theorem in analytic form can be formalized as:

Theorem 2.3. *Let $p : E \to \mathbb{R}$ be a function satisfying:*

$$p(\lambda x) = \lambda p(x) \qquad \forall x \in E \text{ and } \forall \lambda > 0, \tag{3}$$

$$p(x+y) \leq p(x) + p(y) \qquad \forall x, y \in E. \tag{4}$$

Let $G \subset E$ be a linear subspace and let $g : G \to \mathbb{R}$ be a linear function such that:

$$g(x) \leq p(x) \qquad \forall x \in G. \tag{5}$$

Under these assumptions, there exists a linear functional f defined on all of E that extends g, i.e., $g(x) = f(x), \forall x \in G$ such that:

$$f(x) \leq p(x) \qquad \forall x \in E. \tag{6}$$

However, in order to apply for this problem, we shall to use it in its elementary separation form. At first, we need some important definitions.

Definition 2.4. An affine *hyperplane* is a subset H of E of the form:

$$H = \{x \in E; f(x) = \alpha\}, \tag{7}$$

where f is a linear functional that does not vanish identically and $\alpha \in \mathbb{R}$ is a given constant.

Proposition 2.5. *The hyperplane $H = [f = \alpha]$ is closed if and only if f is continuous.*

Definition 2.6. Let A and B be two subsets of E. We say that the hyperplane $H = [f = \alpha]$ separates A and B if

$$f(x) \leq \alpha \quad \forall x \in A \text{ and } f(x) \geq \alpha \quad \forall x \in B. \tag{8}$$

And, we say that H strictly separates A and B if there exists some $\varepsilon > 0$ such that:

$$f(x) \leq \alpha - \varepsilon \quad \forall x \in A \text{ and } f(x) \geq \alpha + \varepsilon \quad \forall x \in B. \tag{9}$$

In particularly, the Hahn-Banach theorem in geometric form can be formalized as:

Theorem 2.7. *Let $A \subset E$ and $B \subset E$ be two nonempty convex subsets such that $A \cap B = \emptyset$. Assume that one of them is open. Then there exists a closed hyperplane that separates A and B.*

The following consequence is the form that we shall use in the proof of the Krein–Milman theorem.

Proposition 2.8. *Let $C \subset E$ be a closed convex set and let $x_0 \in E \setminus C$. Then there exists $f \in E^*$ such that:*

$$\sup_{x \in C} \langle f, x \rangle < \langle f, x_0 \rangle. \tag{10}$$

Proof. Since C is closed and $x_0 \notin C$, we have $d(x_0, C) > 0$. Choose r with $0 < r < d(x_0, C)$ and set $B = B(x_0, r)$. Then B is a nonempty open convex set and $B \cap C = \emptyset$. By the geometric Hahn–Banach theorem, there exist a continuous linear functional $f \in E^*$ and $\alpha \in \mathbb{R}$ such that:

$$\langle f, x\rangle \leq \alpha \quad \forall x \in C, \qquad \langle f, y\rangle \geq \alpha \quad \forall y \in B.$$

Because B is open and contains x_0, there exists $u \in E$ such that $x_0 + u \in B$ and $x_0 - u \in B$. Hence:

$$\langle f, x_0\rangle - |\langle f, u\rangle| \geq \alpha.$$

In particular, $\langle f, x_0\rangle \geq \alpha$. If equality held, then $\langle f, u\rangle = 0$ for every sufficiently small u, which would force $f = 0$, contradicting the fact that the separating hyperplane is nontrivial. Therefore, $\langle f, x_0\rangle > \alpha$, and thus:

$$\sup_{x \in C} \langle f, x\rangle \leq \alpha < \langle f, x_0\rangle.$$

This proves the claim. □

In other words, if $x \neq y$ in a normed vector space, then there exists $f \in E^*$ such that $\langle f, x\rangle \neq \langle f, y\rangle$. This follows by defining a continuous linear functional on $\operatorname{span}\{x - y\}$ and extending it to all of E.

We denote by $\operatorname{ext}(K)$ the set of all extreme points of K and by $\operatorname{co}(K)$ the convex hull of K.

3. Extreme Points and Extreme Sets

Proposition 3.1. *Let $a \in K$. Then a is an extreme point of K if and only if $K \setminus \{a\}$ is convex.*

Proof. Assume first that a is an extreme point, and let $x, y \in K \setminus \{a\}$ with $t \in (0, 1)$. Since K is convex, $tx + (1 - t)y \in K$. If this convex combination were equal to a, then the extremality of a would force $x = y = a$, which is impossible. Hence: $tx + (1 - t)y \in K \setminus \{a\}$, so $K \setminus \{a\}$ is convex.

Conversely, assume that $K \setminus \{a\}$ is convex, and suppose $a = tx + (1 - t)y$ with $t \in (0, 1)$ and $x, y \in K$. If $x \neq a$ and $y \neq a$, then $x, y \in K \setminus \{a\}$, hence the convexity of $K \setminus \{a\}$ implies $a = tx + (1 - t)y \in K \setminus \{a\}$, a contradiction. Therefore, at least one of x, y is equal to a, and then the identity $a = tx + (1 - t)y$ with $t \in (0, 1)$ forces the other one to be equal to a as well. Thus, a is extreme. □

Proposition 3.2. *Let a be an extreme point of K. Let $x_1, \ldots, x_n \in K$ and let $\alpha_1, \ldots, \alpha_n > 0$ satisfy $\sum_{i=1}^n \alpha_i = 1$ and $\sum_{i=1}^n \alpha_i x_i = a$. Then $x_i = a$ for every i.*

Proof. We argue by induction on n. The case $n = 2$ is exactly the definition of an extreme point. Assume the statement true for $n - 1$ terms and write

$$a = \alpha_1 x_1 + (1 - \alpha_1) z, \qquad z = \sum_{i=2}^{n} \frac{\alpha_i}{1 - \alpha_1} x_i \in K.$$

Since a is extreme and $\alpha_1 \in (0, 1)$, we obtain $x_1 = a$ and $z = a$. The induction hypothesis applied to the representation of z then yields $x_i = a$ for $i = 2, \ldots, n$. □

Proposition 3.3. *Let $a \in K$, and assume now that K is compact and convex. Then a is an extreme point of K if and only if $\{a\}$ is an extreme set of K.*

Proof. If a is an extreme point, then $\{a\}$ is nonempty and closed. If $tx+(1-t)y \in \{a\}$ for some $t \in (0,1)$ and $x, y \in K$, then $a = tx + (1-t)y$, hence $x = y = a$. Therefore, $\{a\}$ is an extreme set.

Conversely, if $\{a\}$ is an extreme set and $a = tx + (1-t)y$ with $t \in (0,1)$ and $x, y \in K$, then $tx+(1-t)y \in \{a\}$, so the definition of an extreme set gives $x, y \in \{a\}$. Hence, $x = y = a$, and a is extreme. ☐

4. Every Extreme Set Contains an Extreme Point

From now on, $K \subset E$ is assumed to be nonempty, compact, and convex.

Proposition 4.1. *Let $A \subset K$ be an extreme set and let $f \in E^*$. Define*

$$B = \left\{ x \in A : \langle f, x\rangle = \max_{y\in A} \langle f, y\rangle \right\}. \tag{11}$$

Then, B is an extreme set of K.

Proof. Because A is compact and f is continuous, the maximum is attained, so B is nonempty. Since $B = A \cap f^{-1}(\{\max_A f\})$, it is closed.

Let $t \in (0,1)$ and suppose $tx + (1-t)y \in B$ for some $x, y \in K$. Since $B \subset A$ and A is an extreme set, we first obtain $x, y \in A$. Let $m = \max_{z\in A}\langle f, z\rangle$. Then, we have:

$$m = \langle f, tx + (1-t)y\rangle = t\langle f, x\rangle + (1-t)\langle f, y\rangle \leq tm + (1-t)m = m.$$

Hence, the equality must hold in both inequalities, so $\langle f, x\rangle = \langle f, y\rangle = m$, which means $x, y \in B$. Thus, B is an extreme set. ☐

Proposition 4.2. *Let $M \subset K$ be an extreme set. Consider the family $\mathcal{F}$ of all extreme sets of K contained in M, ordered by:*

$$A \leq B \iff B \subset A. \tag{12}$$

Then $\mathcal{F}$ has a maximal element.

Proof. The set $\mathcal{F}$ is nonempty because $M \in \mathcal{F}$. Let $\mathcal{C} \subset \mathcal{F}$ be a chain. Because the order is reverse inclusion, the members of $\mathcal{C}$ are nested by inclusion. Set:

$$A_{\mathcal{C}} = \bigcap_{A\in\mathcal{C}} A.$$

Each $A \in \mathcal{C}$ is a nonempty closed subset of the compact set M, and the chain condition implies the finite intersection property. Therefore, $A_{\mathcal{C}}$ is nonempty and closed. We claim that it is an extreme set. Indeed, if $tx + (1-t)y \in A_{\mathcal{C}}$ with $t \in (0,1)$ and $x, y \in K$, then $tx + (1-t)y \in A$ for every $A \in \mathcal{C}$, so the extremality of each A yields $x, y \in A$ for every $A \in \mathcal{C}$. Thus, $x, y \in A_{\mathcal{C}}$.

Since $A_{\mathcal{C}} \subset A$ for all $A \in \mathcal{C}$, we have $A \leq A_{\mathcal{C}}$ for all $A \in \mathcal{C}$. Thus, every chain has an upper bound, and Zorn's lemma 2.1 gives a maximal element $M_0 \in \mathcal{F}$. ☐

Proposition 4.3. *The maximal element M_0 furnished by Proposition 4.2 is reduced to a single point.*

Proof. Assume, by contradiction, that M_0 contains two distinct points x and y. By the Hahn–Banach theorem there exists $f \in E^*$ such that $\langle f, x\rangle \neq \langle f, y\rangle$, so f is not constant on M_0. Let

$$B = \left\{ z \in M_0 : \langle f, z\rangle = \max_{w\in M_0} \langle f, w\rangle \right\}.$$

By Proposition 4.1, B is an extreme set of K contained in M_0.

Moreover B is a proper subset of M_0, because f is not constant on M_0. Therefore, $M_0 < B$ in the reverse-inclusion order, contradicting the maximality of M_0. Hence M_0 must be a singleton. □

Corollary 4.4. *Every nonempty extreme set of K contains at least one extreme point of K.*

Proof. Let $M \subset K$ be a nonempty extreme set. By Proposition 4.2, the family of extreme subsets of K contained in M has a maximal element M_0, and by Proposition 4.3 we may write $M_0 = \{a\}$. Proposition 3.3 then shows that a is an extreme point of K. □

5. The Krein–Milman Theorem

Theorem 5.1 (Krein–Milman for compact convex subsets of normed spaces)**.** *Let K be a nonempty compact convex subset of E. Then*

$$K = \overline{\mathrm{co}}(\operatorname{ext} K). \tag{13}$$

Proof. Set $C = \overline{\mathrm{co}}(\operatorname{ext} K)$. By definition, C is a closed convex subset of K. Suppose that $C \neq K$, and choose $x_0 \in K \setminus C$. By Proposition 2.8, there exists $f \in E^*$ such that:

$$\sup_{x \in C} \langle f, x \rangle < \langle f, x_0 \rangle.$$

Let

$$A = \left\{ x \in K : \langle f, x \rangle = \max_{y \in K} \langle f, y \rangle \right\}.$$

By Proposition 4.1, A is an extreme set of K. Therefore, Corollary 4.4 yields an extreme point $a \in A$.

Since $a \in \operatorname{ext} K \subset C$, we have

$$\langle f, a \rangle \leq \sup_{x \in C} \langle f, x \rangle < \langle f, x_0 \rangle.$$

On the other hand, $a \in A$ means that $\langle f, a \rangle = \max_{y \in K} \langle f, y \rangle \geq \langle f, x_0 \rangle$, a contradiction. Hence $C = K$. □

Remark 5.2. The theorem is usually stated in the larger framework of locally convex spaces. In the present problem, the normed-space setting already contains the essential idea: compactness gives supporting faces, and Hahn–Banach supplies the separating functionals.

6. Extreme Points of Unit Balls in Classical Spaces

We now determine the set of extreme points of the closed unit ball B_E in the six spaces listed in Brezis' problem.

Proposition 6.1. *The extreme points of the closed unit ball of ℓ^∞ are exactly the sequences $x = (x_n)$ such that $|x_n| = 1$ for every n.*

Proof. Assume first that $x \in B_{\ell^\infty}$ and that $|x_k| < 1$ for some index k. Choose $\varepsilon > 0$ such that $|x_k| + \varepsilon \leq 1$, and define

$$y = x + \varepsilon e_k, \qquad z = x - \varepsilon e_k.$$

Then $y \neq z$, $x = \frac{1}{2}(y + z)$, and $\|y\|_\infty, \|z\|_\infty \leq 1$, so x is not extreme.

Conversely, assume that $|x_n| = 1$ for every n and write $x = \frac{1}{2}(y + z)$ with $\|y\|_\infty, \|z\|_\infty \leq 1$. For each n we have

$$2x_n = y_n + z_n.$$

If $x_n = 1$, then $y_n + z_n = 2$ with $y_n, z_n \leq 1$, hence $y_n = z_n = 1$. Similarly, if $x_n = -1$, then $y_n = z_n = -1$. Thus, $y = z = x$, and x is extreme. □

Proposition 6.2. *The extreme points of the closed unit ball of c are exactly the convergent sign sequences, that is,*

$$\operatorname{ext}(B_c) = \{x = (x_n) \in c : x_n \in \{-1, 1\} \text{ for all } n\}. \tag{14}$$

Equivalently, these are the $\{-1, 1\}$-valued sequences that are eventually constant.

Proof. As a closed subspace of ℓ^∞, the space c inherits the same perturbation argument as in Proposition 6.1. Thus, any extreme point must satisfy $|x_n| = 1$ for all n.

Conversely, if $x \in c$ is such a sequence and $x = \frac{1}{2}(y + z)$ with $y, z \in B_c$, the coordinate-wise argument from Proposition 6.1 again gives $y = z = x$. Hence x is extreme. Since a convergent sequence taking only the values ±1 must eventually be constant, the final description follows. □

Proposition 6.3. *The closed unit ball of c_0 has no extreme points.*

Proof. Let $x \in B_{c_0}$. Because $x_n \to 0$, there exists some index k such that $|x_k| < 1$. The perturbation argument used in Proposition 6.1 then shows that x is the midpoint of two distinct points of B_{c_0}. Therefore, x is not extreme. Since x was arbitrary, so that $\operatorname{ext}(B_{c_0}) = \varnothing$. □

Proposition 6.4. *The extreme points of the closed unit ball of ℓ^1 are exactly the vectors $\pm e_n$, $n \geq 1$.*

Proof. We first prove that each $\pm e_n$ is extreme. Suppose that $e_n = \frac{1}{2}(y + z)$ with $\|y\|_1, \|z\|_1 \leq 1$. Then, we have:

$$1 = \|e_n\|_1 \leq \frac{1}{2}\left(\|y\|_1 + \|z\|_1\right) \leq 1,$$

so equality holds throughout. In particular, $\|y\|_1 = \|z\|_1 = 1$. Since the nth coordinate of e_n is 1, we have $y_n + z_n = 2$ with $|y_n|, |z_n| \leq 1$, hence $y_n = z_n = 1$. The equality $\|y\|_1 = 1$ then forces all other coordinates of y to be zero, and similarly for z. Thus, $y = z = e_n$. The argument for $-e_n$ is identical.

Now let $x \in B_{\ell^1}$ and assume that $x \neq \pm e_n$ for all n. If $\|x\|_1 < 1$, choose n and $\varepsilon > 0$ with $\|x\|_1 + \varepsilon \leq 1$; then x is the midpoint of $x \pm \varepsilon e_n$, both belonging to B_{ℓ^1}. Hence x is not extreme.

It remains to consider the case $\|x\|_1 = 1$. Since x is not one of the vectors $\pm e_n$, there exist distinct indices i, j such that $x_i \neq 0$ and $x_j \neq 0$. Choose $0 < \varepsilon < \min\{|x_i|, |x_j|\}$ and define $h \in \ell^1$ by

$$h_i = \operatorname{sgn}(x_i), \qquad h_j = -\operatorname{sgn}(x_j), \qquad h_k = 0 \text{ for } k \notin \{i, j\}.$$

Then the signs of the ith and jth coordinates do not change in $x \pm \varepsilon h$, and a direct computation gives:

$$\|x \pm \varepsilon h\|_1 = \|x\|_1 = 1.$$

Since $x = \frac{1}{2}((x + \varepsilon h) + (x - \varepsilon h))$ with $x + \varepsilon h \neq x - \varepsilon h$, x is not extreme. So that, $\operatorname{ext}(B_{\ell^1}) = \{\pm e_n : n \geq 1\}$. □

Proposition 6.5. *Let $1 < p < \infty$. The extreme points of the closed unit ball of ℓ^p are exactly the unit sphere:*

$$\operatorname{ext}(B_{\ell^p}) = \{x \in \ell^p : \|x\|_p = 1\}. \tag{15}$$

Proof. If $\|x\|_p < 1$, then x is an interior point of B_{ℓ^p}, hence certainly not extreme.

Conversely, assume $\|x\|_p = 1$ and write $x = \frac{1}{2}(y+z)$ with $\|y\|_p, \|z\|_p \leq 1$. Since the function $t \mapsto |t|^p$ is strictly convex for $1 < p < \infty$, the norm on ℓ^p is strictly convex. Therefore, the equality in:

$$1 = \|x\|_p \leq \frac{1}{2}\|y\|_p + \frac{1}{2}\|z\|_p \leq 1$$

forces $y = z = x$. Thus, every point of the unit sphere is extreme. $\square$

Proposition 6.6. *The closed unit ball of $L^1(\mathbb{R})$ has no extreme points.*

Proof. Let $f \in B_{L^1(\mathbb{R})}$. If $\|f\|_1 < 1$, then f is an interior point of the unit ball, so it is not extreme.

Assume now that $\|f\|_1 = 1$. Consider the finite measure:

$$\nu(A) = \int_A |f(x)|\, dx.$$

Because Lebesgue measure is non-atomic[1], so is ν. Hence there exists a measurable set A such that $\nu(A) = \frac{1}{2}$. Define:

$$g = 2f\mathbf{1}_A, \qquad h = 2f\mathbf{1}_{\mathbb{R}\setminus A}.$$

Then

$$\|g\|_1 = 2\nu(A) = 1, \qquad \|h\|_1 = 2\nu(\mathbb{R}\setminus A) = 1,$$

and

$$f = \frac{1}{2}(g+h).$$

Since $\nu(A) = \nu(\mathbb{R}\setminus A) = \frac{1}{2}$, both g and h are nonzero, hence g vanishes almost everywhere on $\mathbb{R}\setminus A$ while h vanishes almost everywhere on A. Since both A and $\mathbb{R}\setminus A$ have positive ν-measure, neither of these two functions can agree almost everywhere with the other. Thus $g \neq h$ in $L^1(\mathbb{R})$. Therefore, f is not extreme. We conclude that $\operatorname{ext}(B_{L^1(\mathbb{R})}) = \varnothing$. $\square$

7. CONCLUSION

Problem 1 packages the proof of the Krein–Milman theorem into a sequence of elementary but revealing reductions. The central mechanism is the passage from a compact convex set to a smaller support face on which a separating functional is maximized. Once one knows that every nonempty extreme subset contains an extreme point, the final separation argument becomes natural and transparent. The examples of unit balls show that this abstract theorem is not merely existential: the geometry of extreme points can vary from a rich family of sign sequences in ℓ^∞ to complete absence of extreme points in c_0 and $L^1(\mathbb{R})$.

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[1] The Lebesgue measure meaning no single point, nor any countable set of points, has a positive measure. The measure of any single point is zero and the measure can take any real value in the range $[0, \infty]$

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